Structure and representation theory of double group of four-dimensional cubic group

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Abstract

We generalize the concept of cubic group into any dimension and derive their conjugate classifications and representation theorys. Double group and spinor representation are defined. A detailed calculation is carried out on the structures of four-dimensional cubic group O_4 and its double group, as well as all inequivalent single-valued representations and spinor representations of O_4. All representations are derived adopting Clifford theory of decomposition of induced representations.

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I Introduction

It is well-known that electrons stay in spinor representations of symmetry group of a given lattice in condensed matter physics; it is reasonable to assume that quarks, leptons, as well as baryons, should reside in spinor representations of symmetry group of a four-dimensional lattice in lattice field theory (we will make the concept "spinor representation" precise in the next section). Accordingly, to explore the structure and representations (spinor representations especially) of such groups has important significance in high energy physics.

In the following we concentrate on hyper-cubic case, though it is not the maximum symmetric lattice in four dimension [1]. The first representation-theoretical consideration of symmetry group of such lattice was given by A. Young [2]. Then mathematicians worked in this field due to the interest of wreath product [3][4] to which A. Kerber gave a thorough review in his book [5]. Physicists took part in after K. G. Wilson introduced lattice gauge theory [6]. M. Baake *et al* first gave an explicit description of characters of four-dimensional cubic group [7]; J. E. Mandula *et al* derived the same results using a different method [8]. However, the problem of spinor representations is still a vacancy.

In this paper, we make full use of the power of Clifford theory on decomposition of induced representations (Sec.II.1). We give a systematic and schematic description of conjugate classification and representation theory of generalized cubic group O_n (Sec.II.2). The concept double group is introduced in Sec.II.3 to clarify the terms "single-valued representation" and "two-valued representation(spinor representation)". Then we specify our general results to four dimension, give a detailed description of structure and conjugate classification of O_4 and its double $\overline{O_4}$ (Sec.III), and derive all inequivalent single-valued representations as well as spinor representations of O_4, adopting Clifford theory (Sec.IV). We would like point out that the "spinor" part of our work is completely new and that although the "single-valued" part is well-known, our method to derive them is much more tidy and systematic than that used by other authors who gave the same results, thanks for the power of Clifford theory.

II Conceptual foundations

II.1 Clifford theory on decomposition of induced representations

We will apply two results of Clifford theory, a powerful method in decomposing induced representations of a given group E with a normal subgroup N [9][10][11][12]. We will use $\mathcal{C}[E]$ for the group algebra of E in complex field and G for E/N below. The first result is

Theorem 1 *(Clifford)[13][9] Let M be a simple $\mathcal{C}[E]$-module, and L a simple $\mathcal{C}[N]$-submodule of M_N s.t. L is stable relative to E, i.e. L is isomorphic to all of its conjugates. Then*

$$M \cong L \otimes_{\mathcal{C}} I$$

for a left ideal I in $End_{\mathcal{C}[E]} L^E$. The E-action on $L \otimes_{\mathcal{C}} I$ is given by

$$x \mapsto U(x) \otimes V(x), x \in E$$

where $U : E \to GL(L)$ is a projective representation of E on L, and $V : E \to GL(I)$ is a projective representation of G, that is, $V(x)$ depends only on the coset xN of x in G, for each $x \in E$. The factor sets associated with U and V are inverse of each other.

The second result can be viewed as a special case of Theorem 1. Let $E = N \rtimes G, |E| < \infty$ and N be abelian, then adjoint action of G upon N makes N a G-module. This G-action can be extended naturally to a G-action upon $\mathcal{C}[N]$ by linearity. Define $\Pi(N) := \{\pi_\mu\} \subset \mathcal{C}[N]$,

$$\pi_\mu := \sum_{a\in N} \chi_\mu(a^{-1})a \tag{1}$$

where χ_μ are all irreducible representations of N. The G-action on $\Pi(N)$ is closed and thus $\Pi(N)$ is separated into orbits $\Pi(N) = \coprod_{o\in\mathcal{I}} \Pi_o$ where $\mathcal{I}$ is a index set to label different orbits. For each Π_o, choose one of its element and denote it as $\pi_{o,e}$. The stablizer of each $\pi_{o,e}$ in G (*little group*) is denoted as S_o. There is a bijection from $G/S_o = \{hS_o\}$ to Π_o defined by

$$Ad_h(\pi_{o,e}) = h\pi_{o,e}h^{-1} =: \pi_{o,h} \tag{2}$$

where $\{h\}$ is a system of representatives of left cosets G/S_o. Define

$$\Pi_{o,h;\eta,i} \equiv \pi_{o,h}h \otimes_{S_o} e^o_{\eta,i} \tag{3}$$

in which $\{e^o_{\eta,i}|i = 1, 2, ..., d^o_\eta\}$ with fixed o, η is the ηth irreducible representation of S_o whose dimension is d^o_η, then

Proposition 1 *(little group method)[10][11][12]*

1. *For each fixed (o, η), $\{\Pi_{o,h;\eta,i}\}$ induces an irreducible representation of E, denoted as $D_{o,\eta}$;*
2. *If $(o, \eta) \neq (o', \eta')$, then $D_{o,\eta}$ and $D_{o',\eta'}$ are inequivalent;*
3. *$\{D_{o,\eta}\}$ gives all inequivalent irreducible representations of E.*

II.2 Cubic group in any dimension

The symmetry group of a cube including inversions in three dimensional Euclidean space, which is denoted as O_h in the theory of point groups [14], can be generalized into any n-dimensional Euclidean space E^n, along two different approaches whose results are equivalent.

The first approach of generalization which is very natural and straightforward is geometrical. An *n-cube* (or *hyper-cube in E^n*) C_n is defined to be a subset of E^n, $C_n = \{p|x^i(p) = \pm 1\}$, where $x^i : E^n \to \mathcal{R}, i = 1, 2, ..., n$ are coordinate functions of E^n, together with the distance inherited from E^n. *n-Cubic group* (*hyper-cubic group of degree n*) O_n consists of all isometries of E^n which stabilize C_n. While the second approach of generalization is algebraic. O_h has a semi-direct product structure as $Z_2^3 \rtimes S_3$ [10]; we generalized this to $Z_2^n \rtimes S_n$ which is just a *wreath product* $Z_2 \wr S_n$ of Z_2 with S_n. We point out that these two generalizations are identical. Let $\{e_i\}$ be a standard orthogonal basis of E^n, namely $x^j(e_i) = \delta^j_i$. Define n points in C_n to be $p_0 = (-1, -1, ..., -1), p_i = p_0 + 2e_i$.

Lemma 1 $\forall \epsilon \in O_n$, *ϵ is entirely determined by images $\epsilon(p_i), i = 0, 1, 2, ..., n$.*

Proof:
The fact that ϵ is an isometry of E^n ensures the equality of Euclidean distances $d(p, p_i) = d(\epsilon(p), \epsilon(p_i)), i = 0, 1, ..., n$ for any other p in C_n. If all $\epsilon(p_i)$ are given, $\epsilon(p)$ will be fixed for any other p accordingly due to the fundamental lemma of Euclidean geometry (lemma A-1 in Appendix A). In fact, the existence of solution

in lemma A-1 is guaranteed by that ϵ stablizes C_n and lemma A-1 itself ensures the uniqueness.
□
To fix $\epsilon(p_0)$, there are 2^n ways; while for a fixed $\epsilon(p_0)$, there are $n!$ possibilities to fix $\epsilon(p_i), i = 1, 2, ..., n$. Therefore, $|O_n| = 2^n \cdot n!$.

Proposition 2 *(Structure of O_n)*

$$O_n \cong Z_2^n \rtimes S_n \tag{4}$$

Proof:
Introduce a class of isometries in E^n:

$$\sigma(e_i) = e_{\sigma(i)}; I_i(e_j) = (1 - 2\delta_{ij})e_j, i = 1, 2, ..., n \tag{5}$$

where $\sigma \in S_n$ permute the axes and I_i inverts the ith axis. Subjected to the relations

$$I_i^2 = e, I_iI_j = I_jI_i, i, j = 1, 2, ..., n; \sigma I_i = I_{\sigma(i)}\sigma, \sigma \in S_n \tag{6}$$

these isometries generate a sub-group of C_n isomorphic to $Z_2^n \rtimes S_n$ whose order is $2^n n! = (|O_n|)$. So (4) follows.
□
A. Kerber gave a detailed introduction on the conjugate classification and representation theory of a general wreath product $N \wr G$ in [5]. We specify his general results to our case $Z_2^n \rtimes S_n \cong Z_2 \wr S_n$.

We recall some fundamental facts about symmetrical groups S_n [14]. Each element $\sigma \in S_n$ has a cycle decomposition

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix} = \prod_{k=1}^{n} \prod_{\alpha=1}^{\nu_k} \tau_{k\alpha} \tag{7}$$

where $\tau_{k\alpha}$ are independent k-cycles, which can be expressed as $(a_1a_2...a_k)$, and write $n(k, \alpha) = \{a_1, a_2, ..., a_n\}$. The cycle structure of σ can be represented formally as

$$(\nu) = \prod_{k=1}^{n} (k^{\nu_k}) \tag{8}$$

where $\{\nu_k\}$ satisfies $\sum_{k=1}^{n} k \cdot \nu_k = n$. Two elements in S_n are conjugate equivalent, iff they have the same cycle structure. The number of elements in class (ν) is equal to $N_{(\nu)} = n!/\prod_{k=1}^{n}(k^{\nu_k}\nu_k!)$. Each cycle structure (ν) can be visualized by one unique Young diagram which is denoted also by (ν). There is a one-one correspondence between all inequivalent irreducible representations of S_n and all Young diagrams, which enable us to represent each irreducible representation by the corresponding Young diagram (ν). We write the basis of one of these representations (ν) in $d_{(\nu)}$ dimension as $e_{(\nu)i}, i = 1, 2, ..., d_{(\nu)}$.

We point out that the conjugate classification of O_n has a deep relation with that of S_n. A generic element in $Z_2 \wr S_n$ can be written as

$$\sigma \cdot \prod_i I_i^{s_i} = \begin{pmatrix} 1 & 2 & \dots & n \\ (-)^{s_1}\sigma(1) & (-)^{s_2}\sigma(2) & \dots & (-)^{s_n}\sigma(n) \end{pmatrix} \tag{9}$$

in which $s_i \in \mathcal{Z}/2\mathcal{Z}$. We call the r.h.s. of (9) by *permutation with signature.* $\sigma \prod_i I_i^{s_i}$ can be decomposed according to (7), i.e.$\prod_i I_i^{s_i} = \prod_{k=1}^{n} \prod_{\alpha=1}^{\nu_i} \prod_{a\in n(k,\alpha)} I_a^{s_a}$ and

$$\sigma \prod_i I_i^{s_i} = \prod_{k=1}^{n} \prod_{\alpha=1}^{\nu_i} (\tau_{k\alpha} \prod_{a\in n(k,\alpha)} I_a^{s_a}) \tag{10}$$

The *cycle with signature* is defined to be

$$\tau_{k\alpha} \prod_{a\in n(k,\alpha)} I_a^{s_a} = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ (-)^{s_{a_1}} a_2 & (-)^{s_{a_2}} a_3 & \dots & (-)^{s_{a_k}} a_1 \end{pmatrix}$$

For two independent $(k,\alpha), (k',\alpha')$, it is easy to verify that

$$\tau_{k\alpha}\tau_{k'\alpha'} = \tau_{k'\alpha'}\tau_{k\alpha}, \prod_{a\in n(k,\alpha)} I_a^{s_a}\tau_{k'\alpha'} = \tau_{k'\alpha'} \prod_{a\in n(k,\alpha)} I_a^{s_a}, \prod_{a\in n(k',\alpha')} I_a^{s_a}\tau_{k\alpha} = \tau_{k\alpha} \prod_{a\in n(k',\alpha')} I_a^{s_a}$$

Proposition 3 *[3][4][5]We use $\sim$ to denote conjugate equivalent.*

1. *(descent rule)*

$$\sigma \prod_i I_i^{s_i} \sim \sigma' \prod_i I_i^{s_i'} \Rightarrow \sigma \sim \sigma' \tag{11}$$

2. *(permutation rule) Let*

$$\tilde{\sigma} = \begin{pmatrix} 1 & 2 & \dots & n \\ \tilde{\sigma}(1) & \tilde{\sigma}(2) & \dots & \tilde{\sigma}(n) \end{pmatrix} = \begin{pmatrix} \sigma(1) & \sigma(2) & \dots & \sigma(n) \\ \sigma'(1) & \sigma'(2) & \dots & \sigma'(n) \end{pmatrix}$$

then

$$\tilde{\sigma}(\sigma \prod_i I_i^{s_i})\tilde{\sigma}^{-1} = \begin{pmatrix} \tilde{\sigma}(1) & \tilde{\sigma}(2) & \dots & \tilde{\sigma}(n) \\ (-)^{s_1}\sigma'(1) & (-)^{s_2}\sigma'(2) & \dots & (-)^{s_n}\sigma'(n) \end{pmatrix} \tag{12}$$

3. *(signature rule within one cycle) Let $\tau_{k\alpha}$ be a k-cycle and a_0 be a given number in $n(k,\alpha)$, then*

$$\tau_{k\alpha} \prod_{a\in n(k,\alpha)} I_a^{s_a} \sim \tau_{k\alpha} \prod_{a\in n(k,\alpha)} I_a^{s_a+\delta_{aa_0}+\delta_{a,\tau_{k\alpha}(a_0)}} \tag{13}$$

Note that $\tau_{k\alpha}(a_0)$ is calculated modulo k (the subscripts of I_a are always understood in this way).

4. *(signature rule between two cycles) Let $\tau_{k\alpha}, \tau_{k\beta}$ be two independent k-cycles and we define a bijection $\theta : n(k,\alpha) \to n(k,\beta), a_i \mapsto b_i$. Then*

$$\tau_{k\alpha} \prod_{a\in n(k,\alpha)} I_a^{s_a} \cdot \tau_{k\beta} \prod_{b\in n(k,\beta)} I_b^{s_b} \sim \tau_{k\alpha} \prod_{a\in n(k,\alpha)} I_a^{s_{\theta(a)}} \cdot \tau_{k\beta} \prod_{b\in n(k,\beta)} I_b^{s_{\theta^{-1}(b)}} \tag{14}$$

This theorem ensures conjugate classification of $Z_2 \wr S_n$ is totally determined by the structure of cycles with signature. We verify this statement by generalizing Young diagram technology. First, draw a *Young diagram with numbers and signatures* for each element $\sigma \prod_i I_i^{s_i} \in Z_2 \wr S_n$ according to the decomposition Eq.(10) by the following rules:

1. Plot Young diagram of the class in S_n to which σ belongs and fill each row of this Young diagram with numbers in corresponding cycle by cyclic ordering from up-most box to down-most box.

2. Draw a *slash* in the Young box if the number in this box is mapped to a minus-signed number.

Secondly, partition elements in $Z_2 \wr S_n$ by their cycle structure in S_n and Eq.(11) guarantees elements belong to different partitions can not be conjugate equivalent. Eq.(12) implies that all the numbers that we filled by rule 1 are unnecessary, so smear them out and leave boxes and *slash*es only. Within each row, Eq.(13) says that the positions of *slash*es make no difference. What's more, in fact only that the total number of *slash*es is even or odd distinguishes different classes. Therefore we regulate each row to contain zero or one *slash* at the bottom box. Eq.(14) shows that we can not distinguish the case that one row without any *slash* (Mr. Zero) is put to the left to another row with one *slash* (Mr. One) from that Mr.Zero is to the right of Mr.One, if they have same cyclic length; thus we regulate that Mr.Zero shall always stand left to Mr.One. Therefore, conjugate classes of $Z_2 \wr S_n$ can be uniquely characterized by generalizing Young diagrams containing *slash*es. Following Eq.(8), we represent conjugate classes by

$$(\nu^+, \nu^-) = \prod_{k=1}^{n} (k^{\nu_k^+ + \nu_k^-}) \tag{15}$$

where ν_k^+ is the number of Mr.Zero-type k-cycles and ν_k^- is that of Mr.One-type k-cycles, which satisfy $\nu_k^+ + \nu_k^- = \nu_k$. It is not difficult to check some numerical properties of conjugate classes of $Z_2 \wr S_n$.

Corollary 1 *1. Given a class (ν) in S_n, there are*

$$\prod_{k=1}^{n} (1 + \nu_k) \tag{16}$$

classes in $Z_2 \wr S_n$ which descend to (ν).

2. The number of elements in a class (ν^+, ν^-) is

$$N_{(\nu^+,\nu^-)} = N_{(\nu)} \prod_{k=1}^{n} (C_{\nu_k}^{\nu_k^+} (\sum_{i=0}^{[\frac{k}{2}]} C_k^{2i})^{\nu_k^+} (\sum_{j=1}^{[\frac{k+1}{2}]} C_k^{2j-1})^{\nu_k^-}) \tag{17}$$

where C_m^n is combinatorial number defined to be $m!/(n!(m-n)!)$.

3. The order of a class (ν^+, ν^-) is

$$lcm(\{k \cdot 2^{\delta(\nu_k^-)} | \nu_k \neq 0\}) \tag{18}$$

where $\delta(\nu_k^-) = 0$, if $\nu_k^- = 0$; $\delta(\nu_k^-) = 1$, if $\nu_k^- > 0$.

4. Determinant (signature, parity) of a class

$$det((\nu^+, \nu^-)) = (-1)^{\sum_{k=1}^{n} \nu_k^-} \cdot det((\nu)) \tag{19}$$

where $det((\nu))$ is the determinant of (ν) in S_n.

All inequivalent irreduciblerepresentations of Z_2^n can be expressed as

$$\chi_{(s)} := \bigotimes_{p=1}^{n} \chi_{(-)^{s_p}} \tag{20}$$

in which $s_p \in \mathcal{Z}/2\mathcal{Z}, p = 1, 2, ..., n$ and $\chi_{(-)}, \chi_{(+)}$ are two irreducible representations of Z_2 with $\chi_{(+)}$ being the unit representation. Thus $\pi_{(s)}$ can be defined by Eq.(1) and $\Pi(Z_2^n) = \{\pi_{(s)}\}$. [a)] $\Pi(Z_2^n)$ is divided into $n+1$ orbits under the S_n-action, namely $\Pi(Z_2^n) = \coprod_{p=0}^{n} \Pi_p$. For a given p, Π_p consists of those $\pi_{(s)}$ who has p components in (s) equal to 1, other $n-p$ components equal to 0; hence $|\Pi_p| = C_n^p$. Each $\pi_{p,e}$ is specified to a $\pi_{(s)}$ with $s_p = 0, p = 1, 2, ..., n-p; s_p = 1, p = n-p+1, ..., n$, whose stationary subgroup is just $S_{(n-p)} \otimes S_p$, denoted as F_p. Representatives of left-cosets in S_n/F_p are written as σ_r. , then according to Eqs.(2)(3) and Theorem 1,

Proposition 4 *(Representation theory of $Z_2 \wr S_n$)*

$$\Pi_{p,\sigma_r;(\mu)i,(\nu)j} = \pi_{p,\sigma_r}\sigma_r \otimes_{F_p} (e_{(\mu)i} \otimes e_{(\nu)j})$$

give all inequivalent irreducible representations of $Z_2 \wr S_n$ when $(p, (\mu), (\nu))$ runs over its domain.

where $\pi_{p,\sigma_r} = Ad_{\sigma_r}(\pi_{p,e})$ whose (s) will be denoted as $(s^{(p\sigma_r)})$.

Corollary 2 *1. (Burside formula)* $\sum_{(p,(\mu),(\nu))} (C_n^p d_{(\mu)} d_{(\nu)})^2 = 2^n n!$

2. The number of conjugate classes is $\sum_{(p,(\mu),(\nu))} 1$.

3. (Representation matrix element) Given $\sigma \prod_q I_q^{t_q} \in Z_2 \wr S_n$,

$$D_{(p,(\mu),(\nu))}(\sigma \prod_q I_q^{t_q})_{\sigma_r ij}^{\sigma'_r i' j'} = \delta_{\tilde{\sigma_r}(\sigma\sigma_r)}^{\sigma'} D_{(\mu)}(\sigma_{(n-p)}(\sigma\sigma_r))_i^{i'} D_{(\nu)}(\sigma_p(\sigma\sigma_r))_j^{j'} \prod_q (-)^{s_q^{(p\sigma_r)} t_q}$$

4. (Character)

$$\chi_{(p,(\mu),(\nu))}(\sigma \prod_q I_q^{t_q}) = \delta_{\tilde{\sigma_r}(\sigma\sigma_r)}^{\sigma'} \chi_{(\mu)}(\sigma_{(n-p)}(\sigma\sigma_r)) \chi_{(\nu)}(\sigma_p(\sigma\sigma_r)) \prod_q (-)^{s_q^{(p\sigma_r)} t_q}$$

where $\tilde{\sigma_r}, \sigma_{(n-p)}, \sigma_p$ map an element in S_n to its decompositions according to S_n/F_p, S_{n-p} and S_p respectively.

II.3 Double group and spinor representation

Some fundamental facts of Clifford algebra are necessary for giving the definition and properties of double groups. Denote the Clifford algebra upon Euclidean space V as $Cl(V)$; the isometry $x \mapsto -x$ on V extends to an automorphism of $Cl(V)$ denoted by $x \mapsto \tilde{x}$ and referred to as the canonical automorphism of $Cl(V)$. We use $Cl^*(V)$ to denote the multiplicative group of invertible elements in $Cl(V)$ and the Pin group is the subgroup of $Cl^*(V)$ generated by unit vectors in V , i.e.

$$Pin(V) := \{a \in Cl^*(V) : a = u_1 \cdots u_r, u_j \in V, \|u_j\| = 1\}$$

Proofs of the following four statements can be found in [15].

[a)] $\pi_{(s)}$ satisfy $\pi_{(s)}\pi_{(s')} = \pi_{(s\cdot s')}$ where $(s \cdot s')(p) = s(p)s'(p)$.

Lemma 2 *If $u \in V$ is nonnull, then R_u, reflection along u, is given in terms of Clifford multiplication by*

$$R_u x = -uxu^{-1}, \forall x \in V$$

Theorem 2 *The sequence*

$$0 \to Z_2 \to Pin(V) \overset{\widetilde{Ad}}{\to} O(V) \to 1$$

is exact, in which

$$\widetilde{Ad}_a(x) := \tilde{a}xa^{-1}, \forall x \in Cl(V), a \in Pin(V)$$

We will usually write $\widetilde{Ad}$ just by π as a surjective homomorphism.

Proposition 5 *$Cl(E^4)$, as an associative algebra with unit, is isomorphic to $M_2(\mathbf{H})$ where $\mathbf{H}$ denotes quaternions.*

Lemma 3 *Under the above algebra isomorphism, the image of $Pin(E^4)$ is a subset of $SU(4)$.*

Now we give the main definition of this paper.

Definition 1 *Let ϵ be an injective homomorphism from a group G to $O(n)$, then the double group or the spin-extension of G with respect to ϵ is defined to be $D_n(G, \epsilon) := \pi^{-1}(G)$.*

An introduction to double groups in three dimension can be found in [16]. Following elementary facts in the theory of group extension [17], this diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_2 & \longrightarrow & Pin(E^n) & \xrightarrow{\pi} & O(n) & \longrightarrow & 1 \\
 & & \Vert & & \uparrow & & \uparrow \epsilon & & \\
0 & \longrightarrow & Z_2 & \longrightarrow & \pi^{-1}(G) & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

is commutative. If $(G, \epsilon_1) \sim (G, \epsilon)$, there is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_2 & \xrightarrow{i} & \pi^{-1}((G_1, \epsilon_1)) & \xrightarrow{\pi} & (G, \epsilon_1) & \longrightarrow & 1 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_2 & \xrightarrow{i'} & \pi^{-1}((G_2, \epsilon_2)) & \xrightarrow{\pi'} & (G, \epsilon_2) & \longrightarrow & 1
\end{array}$$

Note that the double group is not a universal object for a given abstract group G but a special type of Z_2-central extension of G subjected to the embedding ϵ. For example, the results of doubling two Z_2 subgroups in $O(2)$, $I := \{1, \sigma\}, R := \{1, R(\pi)\}$ where σ denotes reflection along y-axes and $R(\pi)$ is the rotation over π, are $\pi^{-1}(I) \cong Z_2 \otimes Z_2$ while $\pi^{-1}(R) \cong Z_4$. Nevertheless, we will use symbol $\bar{G}$ to denote the double group at most cases where n and ϵ are fixed. Meanwhile, symbol $\bar{e}$ is adopted to refer -1 in Clifford algebra and is called *central element.*

Let $s : G \to \bar{G}, s.t. \pi s = Id_G$, namely s is a cross-section of π. There is a property of the conjugate classes of $\bar{G}$ which is easy to verify.

Lemma 4 *Let C be a conjugate class in G, then either will $\pi^{-1}(C)$ be one conjugate class in $\bar{G}$ satisfying $\forall g \in C, s(g) \sim -s(g)$; or it will split into two conjugate classes C_1, C_2 in $\bar{G}$ s.t. $\forall g \in C, s(g) \in C_1 \Leftrightarrow -s(g) \in C_2$.*

We will give a more deep result on the splitting of conjugate classes when doubling G to $\bar{G}$ in our another paper.

Let r be an irreducible representation of $\bar{G}$ on V, then $r(-1) = \pm\mathbf{1}$.

Definition 2 *An irreducible representation of $\bar{G}$ with $r(-1) = \mathbf{1}$ is called a single-valued representation of G while an irreducible representation with $r(-1) = -\mathbf{1}$ is called a spinor representation or two-valued representation of G.*

Proposition 6 *Let $IRR_{\mathcal{C}}(G)$ be the class of all inequivalent irreducible representations of G and $IRR_{\mathcal{C}}(G)^s$ be the class of all inequivalent single-valued representations of G, define $\phi : IRR_{\mathcal{C}}(G) \to IRR_{\mathcal{C}}(G)^s, r \mapsto r\circ\pi$. Then ϕ is a bijection.*

Proof:
One can check: $r \circ \pi$ is a representation of $\bar{G}$; if $r \cong r'$, then $r \circ \pi \cong r' \circ \pi$; that r is irreducible implies that $r \circ \pi$ is irreducible and $r \circ \pi$ is single-valued. Therefore, ϕ is well-defined. If r and r' are inequivalent, then $r \circ \pi$ and $r' \circ \pi$ are two elements in $IRR_{\mathcal{C}}(G)^s$, namely ϕ is injective. To prove that ϕ is a surjection, consider any $\tilde{r} \in IRR_{\mathcal{C}}(G)^s : \bar{G} \to V$. Define $r : G \to V, g \mapsto \tilde{r}(s(g))$ where $s(g)$ is any element in $\pi^{-1}(g)$. One can check: r is a well-defined map since $\tilde{r}$ is single-valued; r is an irreducible representation of G on V, accordingly $r \in IRR_{\mathcal{C}}(G)$ and lastly, $\phi(r) = \tilde{r}$. So the result follows.
□

This proposition says that all single-valued representations of G which are part of inequivalent irreducible representations of $\bar{G}$ are completely determined by the representation theory of G.

III Structure of $\overline{O_4}$

III.1 Structure of O_4

Following Proposition 2, $O_4 \cong Z_2^4 \rtimes S_4$; hence $|O_4| = 384$. In point group theory, rotation subgroup of O_h is denoted as O and $S_4 \cong Z_2^2 \rtimes S_3 \cong O$ [10]. We write the isomorphism explicitly. The structure of $Z_2^2 \rtimes S_3$ is given by four generators α, β, η, t and the relations

$$\alpha^2 = e, \beta^2 = e, \alpha\beta = \beta\alpha \tag{21}$$
$$t^3 = e, \eta^2 = e, \eta t = t^2\eta \tag{22}$$
$$t\alpha = \alpha\beta t, t\beta = \alpha t, \eta\alpha = \beta\eta \tag{23}$$

and the isomorphisms are defined to be

$$(12)(34) \leftrightarrow \alpha \leftrightarrow diag(-1,-1,1), (13)(24) \leftrightarrow \beta \leftrightarrow diag(1,-1,-1)$$
$$(234) \leftrightarrow t \leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, (23) \leftrightarrow \eta \leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The structure of $Z_2^4 \rtimes S_4$ is given by (21)(22)(23) together with (see Eq.(6))

$$I_i^2 = e, I_iI_j = I_jI_i, i,j = 1..4, i \neq j \tag{24}$$

$$\begin{aligned} \alpha I_1 = I_2\alpha, \alpha I_3 = I_4\alpha \\ \beta I_1 = I_3\beta, \beta I_2 = I_4\beta \\ tI_1 = I_1 t, tI_2 = I_4 t, tI_3 = I_2 t, tI_4 = I_3 t \\ \eta I_1 = I_1\eta, \eta I_2 = I_3\eta, \eta I_4 = I_4\eta \end{aligned} \tag{25}$$

The matrix form of O_4 is given by (see Eq.(5))

$$(I_I)^j_k = \delta^j_k(1-2\delta^j_i), i,j,k = 1,2,3,4 \tag{26}$$

$$\alpha \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \beta \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} t \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \eta \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{27}$$

In fact, if we introduce

$$\gamma \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the generating relations of O_4 can be reduced to a smaller set $\{I_i, \gamma, t | i = 1,2,3,4\}$ [b)] whose generating relations are (24)(25) together with

$$\gamma^2 = e, t^3 = e, (t\gamma)^4 = e \tag{28}$$

$$\gamma I_1 = I_3\gamma, \gamma I_2 = I_2\gamma, \gamma I_4 = I_4\gamma \tag{29}$$

while $\alpha = (t^2\gamma)^2, \beta = t\gamma t^2\gamma t, \eta = \gamma t\gamma t^2\gamma$.

Applying the general results on conjugate classification of O_n Eqs.(15)(16)...(19), we give the table of conjugate classes of O_4 (see Tab. I).

III.2 Construction of $\overline{O_4}$

We will denote $s(g)$ still as g for all $g \in G$. $\overline{O_4}$ is generated by the equations below.

Proposition 7

$$I_i^2 = -1, I_iI_j = -I_jI_i, i,j = 1..4, i \neq j \tag{30}$$

$$\gamma^2 = -1, t^3 = -1, (t\gamma)^4 = -1 \tag{31}$$

$$\underline{\gamma I_1 = -I_3\gamma}, \gamma I_2 = -I_2\gamma, \gamma I_4 = -I_4\gamma \tag{32}$$

$$tI_1 = I_1t, \underline{tI_2 = I_4t}, \underline{tI_3 = I_2t}, \underline{tI_4 = I_3t} \tag{33}$$

[b)] In fact, we can demand a minimum generator set $\{u, v\}$ which we will not use here.

$$\begin{aligned} u^2 = e, v^6 = e, \gamma = u, t = v^2 \\ (v^2u)^4 = e, (v^3u)^4 = e, u(v^2uv^3uv^4) = (v^2uv^3uv^4)u \\ I_1 = v^3, I_2 = v^2uv^3uv^4, I_3 = uv^3u, I_4 = (v^4uv^3uv^2) \end{aligned}$$

Proof:
First Eqs.(30)...(33) are valid. In fact, the standard orthogonal bases in E^4 satisfy Clifford relations $e_ie_j + e_je_i = -2\delta_{ij}$ which is equivalent to (30); therefore, one can take $I_i = e_i$. Following lemma 2, we set $\gamma = \frac{1}{\sqrt{2}}(e_3 - e_1)$ and check that (32) is satisfied. Let $t = \frac{1}{2}(1 - e_2e_3 + e_2e_4 - e_3e_4)$ which is the product of $\frac{1}{\sqrt{2}}(e_2 - e_3)$ and $\frac{1}{\sqrt{2}}(e_4 - e_2)$, and (33) can be verified. Finally, one can check that (31) is also satisfied.

Second, notice that above equations are just Eqs.(24)(25)(28)(29), which generate O_4, twisted by a Z_2 factor set. So due to the validity of the above equations, $\forall g \in \overline{O_4}$, either g or $-g$ will be generated. But -1 can be generated. Therefore, the above equation set generations $\overline{O_4}$.
□
We can give another proof of this result by proposition 5. In fact, we introduce γ-matrices in E^4 as

$$\gamma_i = \begin{pmatrix} \mathbf{0}_{2\times2} & i\sigma_i \\ i\sigma_i & \mathbf{0}_{2\times2} \end{pmatrix}, i = 1,2,3; \gamma_4 = \begin{pmatrix} \mathbf{0}_{2\times2} & -\mathbf{1}_{2\times2} \\ \mathbf{1}_{2\times2} & \mathbf{0}_{2\times2} \end{pmatrix}$$

in which σ_is stand for three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that our convention here has some difference with the usual one in physics. $\gamma_i (i = 1..4)$ satisfy Clifford relations $\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}\mathbf{1}_{4\times4}$ and $\gamma_i^\dagger = -\gamma_i, \gamma_i\gamma_i^\dagger = \mathbf{1}_{4\times4}, det(\gamma_i) = 1$.

We use $S(g)$ as the image of $s(g)$ in $M_2(\mathbf{H})$. Let

$$S(I_i) = \gamma_i, S(\gamma) = \frac{i}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}, S(t) = \frac{e^{i\frac{7\pi}{4}}}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -i & 1 \end{pmatrix} \tag{34}$$

then one can check that these matrices give correct images under $\widetilde{Ad}$ and satisfy the corresponding relations in (30)...(33). It should be noticed that the $\widetilde{Ad}$-map condition can fix these matrices up to a non-vanishing scalar and that by using lemma 3, the scalar can be fixed up to a Z_4 uncertainty, namely if one find out a $S(g)$ then $iS(g), -S(g), -iS(g)$ will also work. Then one can figure out two of them by calculating the projections on the basis of $Cl(E^4)$ and ruling out those whose projections are pure imagine.

We point out that the generating relations in proposition 7 are not unique, due to the canonical automorphism of $Cl(E^4)$. In fact from the second proof of this proposition, we have notified that at last there is still a Z_2 uncertainty. Consequently, we can change the cross-section s to another one s' by a "local" Z_2 transformation and the underlined equations in Eqs.(30)...(33) may gain or lose some -1-factors accordingly. Anyway, they are equivalent to the former ones.

To classify the elements in $\overline{O_4}$, Lemma 4 will enable us to use the same symbols for the conjugate classes of O_4 and to use a "$\prime$" for those splitting classes. Except for classes $1, 8, 14, 15, 20$ which split into two classes for each, any other class in O_4 is lifted to one class. Therefore, there are totally 25 classes in $\overline{O_4}$ (see table II).

IV Representations of $\overline{O_4}$

IV.1 Single-valued representations of O_4

Due to Theorem 6, there are totally 20 inequivalent single-valued representations of O_4 corresponding to the 20 inequivalent irreducible representations of O_4; the representation theory of O_4 can be systematically solved by applying little group method (Theorem 1).

All inequivalent irreducible characters are listed in Table III. Following Theorem 4, $\Pi(Z_2^4)$ are partitioned into orbits with index set defined in a physical convention $\mathcal{I} := \{S, P, V, A, T\}$.

$$\begin{aligned}
\Pi_S = \{\pi_{0000}\}, F_S \cong S_4; \Pi_P = \{\pi_{1111}\}, F_P \cong S_4;\\
\Pi_V = \{\pi_{0001}, \pi_{0010}, \pi_{0100}, \pi_{1000}\}, F_V \cong S_3;\\
\Pi_A = \{\pi_{1110}, \pi_{1101}, \pi_{1011}, \pi_{0111}\}, F_A \cong S_3;\\
\Pi_T = \{\pi_{0011}, \pi_{0101}, \pi_{1001}, \pi_{0110}, \pi_{1010}, \pi_{1100}\}, F_T \cong Z_2^2
\end{aligned}$$

We will use $[\lambda]$ instead of (ν) to denote Young diagrams where $[\lambda] = [\lambda_1\lambda_2...\lambda_n], \lambda_k = \sum_{i=k}^{n} \nu_i$.

Orbit S

All inequivalent irreducible representations of S_4 is labeled by $[4], [31], [2^2], [21^2], [1^4]$; accordingly,

$$\Pi_S \cdot ([4], [31], [2^2], [21^2], [1^4])$$

provide two one-dimensional, one two-dimensional and two three-dimensional representations. As for representation matrices, all $I_i, i = 1..4$ are mapped to identity, while α, β, t, η take the same matrix form as they have in S_4, i.e. $\Pi_S \cdot [4] : I_i, \alpha, \beta, t, \eta \to 1$; $\Pi_S \cdot [1^4] : I_i, \alpha, \beta, t \to 1, \eta \to -1$;

$$\Pi_S \cdot [2^2] : I_i, \alpha, \beta \to \mathbf{1}_{2\times2}, t \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \eta \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$\Pi_S \cdot [31] : I_i \to \mathbf{1}_{3\times3}, \alpha \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \beta \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, t \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \eta \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$\Pi_S \cdot [21^2]$: I_i, α, β, t take the same form of $\Pi_S \cdot [31]$ and η gains a minus sign compared to $\Pi_S \cdot [31]$.

Orbit P

$$\Pi_P \cdot ([4], [31], [2^2], [21^2], [1^4])$$

The only difference from **Orbit** S is that I_i are mapped to $-\mathbf{1}$.

Orbit V

All inequivalent irreducible representations of S_3 can be written as $[3], [21], [1^3]$ and it has no difficulty, using our generating relations, to check

$$\alpha\pi_{1000}\alpha^{-1} = \pi_{0100}, \eta\pi_{0100}\eta^{-1} = \pi_{0010}, \alpha\pi_{0010}\alpha^{-1} = \pi_{0001}$$

Hence, this orbit gives two four-dimensional representations and one eight-dimensional representation.

$$\Pi_V \cdot ([3],[21],[1^3]) = (e, \alpha, \beta\eta, \alpha\beta\eta) \cdot \pi_{1000} \cdot ([3],[21],[1^3])$$

The representation matrices of $\Pi_V \cdot [3]$ are coincident with those in Eqs.(26)(27). Representation matrices of $\Pi_V \cdot [1^3]$ are the same as those in $\Pi_V \cdot [3]$, except that η picking on a minus sign.

$\Pi_V \cdot [21]$:

$$e_i \to \begin{pmatrix} \Pi_V \cdot [3](e_i) & \mathbf{0}_{4\times4} \\ \mathbf{0}_{4\times4} & \Pi_V \cdot [3](e_i) \end{pmatrix},$$

$$\alpha \to \begin{pmatrix} \Pi_V \cdot [3](\alpha) & \mathbf{0}_{4\times4} \\ \mathbf{0}_{4\times4} & \Pi_V \cdot [3](\alpha) \end{pmatrix}, \beta \to \begin{pmatrix} \mathbf{0}_{4\times4} & \Pi_V \cdot [3](\beta) \\ \Pi_V \cdot [3](\beta) & \mathbf{0}_{4\times4} \end{pmatrix},$$
$$t \to \begin{pmatrix} \omega \cdot \Pi_V \cdot [3](t) & \mathbf{0}_{4\times4} \\ \mathbf{0}_{4\times4} & \omega^2 \cdot \Pi_V \cdot [3](t) \end{pmatrix}, \eta \to \begin{pmatrix} \mathbf{0}_{4\times4} & \Pi_V \cdot [3](\eta) \\ \Pi_V \cdot [3](\eta) & \mathbf{0}_{4\times4} \end{pmatrix}$$

Orbit A
Similar to **Orbit** V, there are two four-dimensional representations and one eight-dimensional representation.

$$\Pi_A \cdot ([3],[21],[1^3]) = (e, \alpha, \beta\eta, \alpha\beta\eta) \cdot \pi_{0111} \cdot ([3],[21],[1^3])$$

while the representation matrices for I_i pick on a minus sign, without changing the others.

Orbit T
The stationary subgroup F_T leaving π_{0110} invariant is $\{e, \eta, \alpha\beta, \alpha\beta\eta\}$ with four one-dimensional irreducible representations, denoted by $\pi_{(a,b)}, a, b = 0, 1$. Therefore, there are four six-dimensional representations given by this orbit. Notice that

$$\alpha\pi_{0110}\alpha^{-1} = \pi_{1001}, t\pi_{1001}t^{-1} = \pi_{1010}, t\pi_{1010}t^{-1} = \pi_{1100},$$
$$\alpha\pi_{1010}\alpha^{-1} = \pi_{0101}, \beta\pi_{1100}\beta^{-1} = \pi_{0011},$$

the four representations can be labeled as

$$\Pi_T \cdot (\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}) = (e, \alpha, \alpha\beta t, \beta t^2, \beta t, t^2) \cdot \pi_{0110} \cdot (\pi_{00} + \pi_{01} + \pi_{10} + \pi_{11})$$

Then we enumerate the matrices for the four representations.

$$\Pi_T \cdot \pi_{00} : I_1 \to diag(1,-1,-1,-1,1,1), I_2 \to diag(-1,1,1,-1,-1,1),$$

$$I_3 \to diag(-1,1,-1,1,1,-1), I_4 \to diag(1,-1,1,1,-1,-1),$$

$$\alpha \to \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \end{pmatrix} \beta \to \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \end{pmatrix}$$

$$t \to \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0 \end{pmatrix} \eta \to \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}$$

$$\Pi_T \cdot \pi_{01} : I_1 \to \Pi_2 \cdot \pi_{00}(I_1), I_2 \to \Pi_T \cdot \pi_{00}(I_2), I_3 \to \Pi_T \cdot \pi_{00}(I_3), I_4 \to \Pi_T \cdot \pi_{00}(I_4)$$

$$\alpha \to \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&-1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&-1 \end{pmatrix} \beta \to \begin{pmatrix} 0&-1&0&0&0&0\\ -1&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&-1&0\\ 0&0&0&1&0&0 \end{pmatrix}$$

$$t \to \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ -1&0&0&0&0&0\\ 0&0&0&0&-1&0 \end{pmatrix} \eta \to \begin{pmatrix} 1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&-1&0 \end{pmatrix}$$

$$\Pi_T \cdot \pi_{10} : I_1 \to \Pi_T \cdot \pi_{00}(I_1), I_2 \to \Pi_T \cdot \pi_{00}(I_2), I_3 \to \Pi_T \cdot \pi_{00}(I_3), I_4 \to \Pi_T \cdot \pi_{00}(I_4)$$

$$\alpha \to \Pi_T \cdot \pi_{00}(\alpha), \beta \to \Pi_T \cdot \pi_{00}(\beta), t \to \Pi_T \cdot \pi_{00}(t), \eta \to (-1) \cdot \Pi_T \cdot \pi_{00}(\eta)$$

$$\Pi_T \cdot \pi_{11} : I_1 \to \Pi_T \cdot \pi_{01}(I_1), I_2 \to \Pi_T \cdot \pi_{01}(I_2), I_3 \to \Pi_T \cdot \pi_{01}(I_3), I_4 \to \Pi_T \cdot \pi_{01}(I_4)$$

$$\alpha \to \Pi_T \cdot \pi_{01}(\alpha), \beta \to \Pi_T \cdot \pi_{01}(\beta), t \to \Pi_T \cdot \pi_{01}(t), \eta \to (-1) \cdot \Pi_T \cdot \pi_{01}(\eta)$$

Here we find all 20 inequivalent irreducible representations corresponding to the 20 conjugate classes of O_4, which satisfy Burside formula

$$2 \times (1^2 + 1^2 + 2^2 + 3^2 + 3^2) + 2 \times (4^2 + 4^2 + 8^2) + 4 \times 6^2 = 384$$

Following proposition 6, we have found all of the single-valued representations of O_4.

IV.2 Spinor representations of O_4

Notice the following facts that $\overline{Z_2^4} \lhd \overline{O_4}$, $\overline{O_4}/\overline{Z_2^4} \cong S_4$ and Eqs.(34) generate a spinor representation of O_4 which is denoted still as S; what's more, its restriction to $\overline{Z_2^4}$ is also a two-valued representation of Z_2^4. These facts ensure two conditions in Theorem 1. To apply Theorem 1 to deduce spinor representations of O_4, we develop a calculation method. The matrices of a spinor representation of O_4 for I_i, γ, t, denoted as $\tilde{S}(I_i), \tilde{S}(\gamma), \tilde{S}(t)$, can be decomposed as

$$\tilde{S}(I_i) = S(I_i) \otimes \mathbf{1}, i = 1, 3, 4; \tilde{S}(I_2) = -S(I_2) \otimes \mathbf{1};$$
$$\tilde{S}(\gamma) = \Gamma \otimes \tilde{\gamma}; \tilde{S}(t) = T \otimes \tilde{t}$$

where Γ, T and $S(I_i)$ act on the same module, $\tilde{\gamma}, \tilde{t}$ have *the same texture (zero matrix elements) of the representation matrices of five inequivalent irreducible representations of* S_4 (the minus added before $S(I_2)$

is for a physical convention). There are five spinor representations of dimension 4,4,8,12 and 12 respectively and the second half of Burside formula is satisfied.

$$4^2 + 4^2 + 8^2 + 12^2 + 12^2 = 384$$

Corresponding the generating equations (30)...(33), there are a system of matrix equations.

$$\tilde{S}(\gamma)^2 = \tilde{S}(t)^3 = -\mathbf{1}, (\tilde{S}(\gamma)\tilde{S}(t))^4 = -\mathbf{1} \tag{35}$$

$$\tilde{S}(\gamma)\tilde{S}(I_2) = -\tilde{S}(I_2)\tilde{S}(\gamma), \tilde{S}(\gamma)\tilde{S}(I_4) = -\tilde{S}(I_4)\tilde{S}(\gamma), \tilde{S}(\gamma)\tilde{S}(I_1) = -\tilde{S}(I_3)\tilde{S}(\gamma), \tag{36}$$

$$\tilde{S}(t)\tilde{S}(I_1) = \tilde{S}(I_1)\tilde{S}(t), \tilde{S}(t)\tilde{S}(I_2) = -\tilde{S}(I_4)\tilde{S}(t), \tilde{S}(t)\tilde{S}(I_3) = -\tilde{S}(I_2)\tilde{S}(t), \tilde{S}(t)\tilde{S}(I_4) = \tilde{S}(I_3)\tilde{S}(t), \tag{37}$$

plus a unitary condition

$$\tilde{S}(\gamma)^\dagger\tilde{S}(\gamma) = \mathbf{1}, \tilde{S}(t)^\dagger\tilde{S}(t) = \mathbf{1} \tag{38}$$

Note that we add a minus sign to the second and the third equations in Eq.(37) compared with Eq.(33) according to the same physics convention, though they are completely equivalent.

Solving Eqs.(35)...(38) for four-dimensional case gives two solutions

$$\underline{4}_+ : \tilde{S}(\gamma) = \Gamma \cdot \tilde{\gamma}_+, \Gamma = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}, \tilde{\gamma}_+ = i$$

$$\tilde{S}(t) = T \cdot \tilde{t}, T = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -i & 1 \end{pmatrix}, \tilde{t} = e^{i\frac{7\pi}{4}}$$

$$\underline{4}_- : \tilde{S}(\gamma) = \Gamma \cdot \tilde{\gamma}_-, \tilde{\gamma}_- = -i, \tilde{S}(t) = T \cdot \tilde{t}$$

Note that $\underline{4}_+$ is just the representation S with $\tilde{S}(I_2) = -S(I_2)$.

As for eight-dimensional case we can suppose

$$\tilde{\gamma} = \begin{pmatrix} 0 & \tilde{c} \\ \tilde{d} & 0 \end{pmatrix}, \tilde{t} = \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix}$$

The solution $\underline{8}$ is given by

$$\tilde{c} = e^{i\frac{\pi}{3}}, \tilde{d} = e^{i\frac{2\pi}{3}}, \tilde{a} = e^{i\frac{5\pi}{12}}, \tilde{d} = e^{i\frac{13\pi}{12}}$$

Finally, we set for the twelve-dimensional case

$$\tilde{\gamma} = \begin{pmatrix} 0 & 0 & \tilde{z} \\ 0 & \tilde{y} & 0 \\ \tilde{x} & 0 & 0 \end{pmatrix}, \tilde{t} = \begin{pmatrix} 0 & 0 & \tilde{n} \\ \tilde{l} & 0 & 0 \\ 0 & \tilde{m} & 0 \end{pmatrix}$$

Such that

$$\underline{12}_+ : \tilde{x} = 1, \tilde{y} = i, \tilde{z} = -1, \tilde{l} = 1, \tilde{m} = 1, \tilde{n} = e^{i\frac{5\pi}{4}}$$

$$\underline{12}_- : \tilde{x} = 1, \tilde{y} = -i, \tilde{z} = -1, \tilde{l} = 1, \tilde{m} = -1, \tilde{n} = e^{i\frac{\pi}{4}}$$

So far, we obtain all inequivalent irreducible representations of $\overline{O_4}$ and we summarize our results in Table IV.

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A Fundamental lemma of n-dimensional Euclidean geometry

Lemma A-1 *(Weak form) Let $p_i, i = 0, 1, 2, ..., n$ be n points in n-dimensional Euclidean space E^n which are non-collinear and given n non-negative real numbers $d_i, i = 0, 1, 2, ..., n$, then there exists at most one point $p \in E^n$ s.t. $d(p, p_i) = d_i$.*

Proof:
Without losing generality, set $p_0 = (0, 0, ..., 0)$ and understand $p, p_i, i = 1, 2, ..., n$ as vectors in E^n. Consider equation set

$$\begin{aligned} (p - p_i, p - p_i) &= d_i^2, i = 1, 2, ..., n && \text{(A-1)} \\ (p, p) &= d_0^2 && \text{(A-2)} \end{aligned}$$

Substitute (A-2) into (A-1)

$$(p_i, p) = \frac{1}{2}(d_0^2 - d_i^2 + (p_i, p_i)), i = 1, 2, ..., n \quad \text{(A-3)}$$

The non-collinearity implies that (A-3) has a solution p. The weak form of *fundamental lemma of Euclidean geometry* follows.
□

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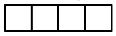
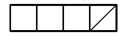
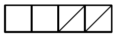
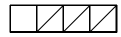
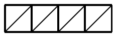
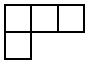
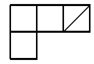
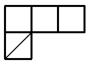
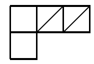
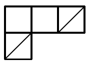
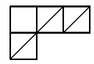
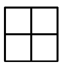
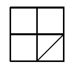
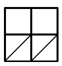
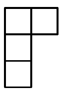
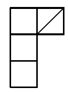
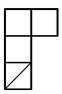
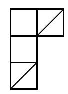
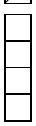
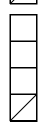
Table I: Conjugate Classes of $Z_2^4 \rtimes S_4$

No	SplitNo	YoungDiagram	ord	num	det
1	1-1		1	1	1
2	1-2		2	4	-1
3	1-3		2	6	1
4	1-4		2	4	-1
5	1-5		2	1	1
6	2-1		2	12	-1
7	2-2		2	24	1
8	2-3		4	12	1
9	2-4		2	12	-1
10	2-5		4	24	-1
11	2-6		4	12	1
12	3-1		2	12	1
13	3-2		4	24	-1
14	3-3		4	12	1
15	4-1		3	32	1
16	4-2		6	32	-1
17	4-3		6	32	-1
18	4-4		6	32	1
19	5-1		4	48	-1
20	5-2		8	48	1

"SplitNo" reflects the relation between the classes of $Z_2 \wr S_4$ and those of S_4. "ord" means order of each class. "num" is the number of elements in each class. "det" is the signature of each class. See Eqs.(16)...(19).

Table II: Conjugate Classes of $\overline{O_4}$

No.	1	1′	2	3	4	5	6	7	8	8′	9	10	11	12	13	14	14′	15	15′	16	17	18	19	20	20′
num.	1	1	8	12	8	2	24	48	12	12	24	48	24	24	48	12	12	32	32	64	64	64	96	48	48
ord.	1	2	4	4	2	2	4	4	8	8	2	8	8	4	8	4	4	6	3	12	6	6	4	8	8

The labels of classes are descended from those of O_4 with "′" for those classes split when lifting into $\overline{O_4}$.

Table III: Character Table of Z_2^4

$\overline{Z_2^4}$	$[e]$	$[I_1]$	$[I_2]$	$[I_3]$	$[I_4]$	$[I_{12}]$	$[I_{13}]$	$[I_{14}]$	$[I_{23}]$	$[I_{24}]$	$[I_{34}]$	$[I_{234}]$	$[I_{134}]$	$[I_{124}]$	$[I_{123}]$	$[I_{1234}]$
χ_{0000}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_{0001}	1	1	1	1	-1	1	1	-1	1	-1	-1	-1	-1	-1	1	-1
χ_{0010}	1	1	1	-1	1	1	-1	1	-1	1	-1	-1	-1	1	-1	-1
χ_{0100}	1	1	-1	1	1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1
χ_{1000}	1	-1	1	1	1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
χ_{0011}	1	1	1	-1	-1	1	-1	-1	-1	-1	1	1	1	-1	-1	1
χ_{0101}	1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
χ_{1001}	1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1
χ_{0110}	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1
χ_{1010}	1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	1
χ_{1100}	1	-1	-1	1	1	1	-1	-1	-1	-1	1	-1	-1	1	1	1
χ_{1110}	1	-1	-1	-1	1	1	1	-1	1	-1	-1	1	1	1	-1	-1
χ_{1101}	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
χ_{1011}	1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	-1	1	1	-1
χ_{0111}	1	1	-1	-1	-1	-1	-1	-1	1	1	1	-1	1	1	1	-1
χ_{1111}	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1

$I_{i_1 i_2 \ldots I_a} := I_{i_1} \cdot I_{i_2} \cdot \ldots \cdot I_{i_a}$. Irreducible characters are labeled as $\chi_{s_1 s_2 s_3 s_4}, s_i \in \mathcal{Z}/2\mathcal{Z}$ (see Eq.(20)).

Table IV: Character Table of $\overline{O_4}$

	Π_S					Π_P					Π_V			Π_A			Π_T				spinor rep.				
	[4]	$[1^4]$	$[2^2]$	[31]	$[21^2]$	[4]	$[1^4]$	$[2^2]$	[31]	$[21^2]$	[3]	$[1^3]$	[21]	[3]	$[1^3]$	[21]	π_{00}	π_{01}	π_{10}	π_{11}	$\underline{4}_+$	$\underline{4}_-$	$\underline{8}$	$\underline{12}_+$	$\underline{12}_-$
1	1	1	2	3	3	1	1	2	3	3	4	4	8	4	4	8	6	6	6	6	4	4	8	12	12
$1'$	1	1	2	3	3	1	1	2	3	3	4	4	8	4	4	8	6	6	6	6	-4	-4	-8	-12	-12
2	1	1	2	3	3	-1	-1	-2	-3	-3	2	2	4	-2	-2	-4	0	0	0	0	0	0	0	0	0
3	1	1	2	3	3	1	1	2	3	3	0	0	0	0	0	0	-2	-2	-2	-2	0	0	0	0	0
4	1	1	2	3	3	-1	-1	-2	-3	-3	-2	-2	-4	2	2	4	0	0	0	0	0	0	0	0	0
5	1	1	2	3	3	1	1	2	3	3	-4	-4	-8	-4	-4	-8	6	6	6	6	0	0	0	0	0
6	1	-1	0	1	-1	1	-1	0	1	-1	2	-2	0	2	-2	0	2	0	-2	0	0	0	0	0	0
7	1	-1	0	1	-1	-1	1	0	-1	1	0	0	0	0	0	0	0	2	0	-2	0	0	0	0	0
8	1	-1	0	1	-1	-1	1	0	-1	1	2	-2	0	-2	2	0	0	-2	0	2	$-2\sqrt{2}$	$2\sqrt{2}$	0	$-2\sqrt{2}$	$2\sqrt{2}$
$8'$	1	-1	0	1	-1	-1	1	0	-1	1	2	-2	0	-2	2	0	0	-2	0	2	$2\sqrt{2}$	$-2\sqrt{2}$	0	$2\sqrt{2}$	$-2\sqrt{2}$
9	1	-1	0	1	-1	1	-1	0	1	-1	-2	2	0	-2	2	0	2	0	-2	0	0	0	0	0	0
10	1	-1	0	1	-1	1	-1	0	1	-1	0	0	0	0	0	0	-2	0	2	0	0	0	0	0	0
11	1	-1	0	1	-1	-1	1	0	-1	1	-2	2	0	2	-2	0	0	-2	0	2	0	0	0	0	0
12	1	1	2	-1	-1	1	1	2	-1	-1	0	0	0	0	0	0	2	-2	2	-2	0	0	0	0	0
13	1	1	2	-1	-1	-1	-1	-2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	1	2	-1	-1	1	1	2	-1	-1	0	0	0	0	0	0	-2	2	-2	2	2	2	4	-2	-2
$14'$	1	1	2	-1	-1	1	1	2	-1	-1	0	0	0	0	0	0	-2	2	-2	2	-2	-2	-4	2	2
15	1	1	-1	0	0	1	1	-1	0	0	1	1	-1	1	1	-1	0	0	0	0	2	2	-2	0	0
$15'$	1	1	-1	0	0	1	1	-1	0	0	1	1	-1	1	1	-1	0	0	0	0	-2	-2	2	0	0
16	1	1	-1	0	0	-1	-1	1	0	0	-1	-1	1	1	1	-1	0	0	0	0	0	0	0	0	0
17	1	1	-1	0	0	-1	-1	1	0	0	1	1	-1	-1	-1	1	0	0	0	0	0	0	0	0	0
18	1	1	-1	0	0	1	1	-1	0	0	-1	-1	1	-1	-1	1	0	0	0	0	0	0	0	0	0
19	1	-1	0	-1	1	1	-1	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	1	-1	0	-1	1	-1	1	0	1	-1	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	$-\sqrt{2}$	$\sqrt{2}$
$20'$	1	-1	0	-1	1	-1	1	0	1	-1	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	0	$\sqrt{2}$	$-\sqrt{2}$